Plane Field Harmonics in Accelerator Magnets

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I. Introduction

The field quality in magnet apertures, or any region that is free of currents and magnetized material, is conveniently described by a set of Fourier coefficients, known as field harmonics or multipole coefficients. The coefficients are determined from a measured field or a field computation on a contour of a suitable coordinate system. They are then compared to the general solution of the Laplace equation, in order to obtain a description of the field in the interior of the contour. Different coordinate systems are chosen to describe fields in cosine-theta magnets, C-magnets, solenoids, bent magnets, and wigglers. The goal of this paper is to show that the underlying methodology is the same in all of the above cases, provided that a sufficiently general approach is adopted that allows to deal with the different representations of the metric. In this abstract, we briefly recall the situation in circular coordinates because they are the most commonly used for the computation of fields in long accelerator magnets. We then show a generalization of the method to elliptic coordinates more suited to magnets where the beam pipe is not circular, e.g., insertion devices for synchrotron light sources.

II. Circular Harmonics

A general solution that satisfies the Laplace equation, $\nabla^2 A_z = 0$, can be found by the separation of variables method. For $A_z = \rho(r)\phi(\varphi)$ we obtain the solutions $\rho_n(r) = \mathcal{A}_n r^n + \mathcal{B}_n r^{-n}$ and $\phi_n(\varphi) = \mathcal{C}_n \sin n\varphi + \mathcal{D}_n \cos n\varphi$. As the vector potential is single-valued, it must be a periodic function in φ with $A_z(r,0) = A_z(r,2\pi)$. The separation constant n takes integer values and therefore the general solution of the Laplace equation is $A_z(r,\varphi) = \sum_{n=1}^{\infty}(\mathcal{A}_n r^n + \mathcal{B}_n r^{-n})(\mathcal{C}_n \sin n\varphi + \mathcal{D}_n \cos n\varphi)$. Let us consider the magnet aperture as the problem domain and denote it Ω_a. The condition that the flux density is finite at $r = 0$ imposes $\mathcal{B}_n = 0$. As the coefficients are not determined at this stage, we save on notation and rewrite the general solution for the vector potential in Ω_a as $A_z(r,\varphi) = \sum_{n=1}^{\infty} r^n(\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi)$. The field components can then be expressed as $B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1}(\mathcal{A}_n \cos n\varphi - \mathcal{B}_n \sin n\varphi)$, and $B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1}(\mathcal{A}_n \sin n\varphi + \mathcal{B}_n \cos n\varphi)$, in Ω_a. Notice the appearance of the metric coefficient $1/r$ in the derivative of the vector potential. Each value of the integer n in the solution of the Laplace equation corresponds to a specific flux distribution generated by ideal magnet geometries. The three lowest values, n = 1,2,3 correspond to the dipole, quadrupole, and sextupole flux density distributions. Assuming that the radial component of the magnetic flux density is measured or calculated at a reference radius $r = r_0$ as a function of the angular position φ, we obtain the Fourier series expansion of the radial field component: $B_r(r_0,\varphi) = \sum_{n=1}^{\infty}(B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi)$, where $A_n(r_0) = \frac{1}{\pi}\int_0^{2\pi} B_r(r_0,\varphi)\cos n\varphi \,\mathrm{d}\varphi$ and $B_n(r_0) = \frac{1}{\pi}\int_0^{2\pi} B_r(r_0,\varphi)\sin n\varphi \,\mathrm{d}\varphi$ for $n = 1,2,3,\ldots$. Comparing the coefficients in the two expressions for the B_r component we obtain $\mathcal{A}_n = \frac{1}{n\, r_0^{n-1}} A_n(r_0)$ and $\mathcal{B}_n = \frac{-1}{n\, r_0^{n-1}} B_n(r_0)$. Thus the field components in the entire domain Ω_a can be expressed as $B_r(r,\varphi) = \sum_{n=1}^{\infty}(r/r_0)^{n-1}(B_n(r_0)\sin n\varphi + A_n(r_0)\cos n\varphi)$ and $B_\varphi(r,\varphi) = \sum_{n=1}^{\infty}(r/r_0)^{n-1}(B_n(r_0)\cos n\varphi - A_n(r_0)\sin n\varphi)$. The normal and skew multipole coefficients $B_n(r_0)$, $A_n(r_0)$ are given in units of tesla at a reference radius r_0, usually chosen to about 2/3 of the magnet aperture [1]. However, the situation is not always that easy because in some coordinate systems the metric coefficient may depend on the angular coordinates as well, which does not allow a direct comparison of coefficients with the Fourier transform. A method to circumvent this problem is presented in the next chapter.

III. Separation in Plane Elliptic Coordinates

Consider a plane ellipse, centered at the origin, with major-semi axis a and minor-semi axis b. A system of elliptic coordinates is defined by the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2 : (\eta,\psi) \mapsto (x,y)$ for $0 \le \eta < \infty$ and $-\pi \le \psi \le \pi$ according to $x = e\cosh\eta\cos\psi$, $y = e\sinh\eta\sin\psi$, where $e = \sqrt{a^2 - b^2}$ is the distance between the origin and the focal points. For the reference ellipse with semi axes $b = e\sinh\eta_0$ and $a = e\cosh\eta_0$ it follows that $\eta_0 = \mathrm{artanh}(b/a)$ for $a > b$. The scale factors for the elliptic coordinates are $h_1 = h_2 = e\sqrt{\cosh^2\eta - \cos^2\psi}$. Written in plane elliptic coordinates, the Laplace equation for the vector potential takes the form: $\nabla^2 A_z = \frac{1}{e^2(\cosh^2\eta - \cos^2\psi)}\left(\frac{\partial^2 A_z}{\partial \eta^2} + \frac{\partial^2 A_z}{\partial \psi^2}\right) = 0$. The separation of variables technique yields for $A_z = H(\eta)\Psi(\psi)$ two ordinary differential equations with the solutions $H_p(\eta) = \mathcal{A}_p\cosh p\eta + \mathcal{B}_p \sinh p\eta$ and $\Psi_p(\psi) = \mathcal{C}_p\cos p\psi + \mathcal{D}_p\sin p\psi$. Since Ψ is a 2π-periodic function, the separation constant p takes only integer values. In [2] it is shown that indeed not all eigenfunctions resulting from the above equations are required. A complete system of orthogonal eigenfunctions is given by $\cos n\psi\cosh n\eta$ and $\sin n\psi\sinh n\eta$ for $n = 1,2,3,\ldots$ Saving again on notation, the general solution can therefore be written as

$$A_z(\eta,\psi) = \sum_{n=1}^{\infty}\left(\mathcal{A}_n \sinh n\eta \sin n\psi + \mathcal{B}_n \cosh n\eta \cos n\psi\right) . \tag{1}$$

The components of the magnetic flux density are calculated from the vector potential by $B_\eta = \frac{1}{h_2}\frac{\partial A_z}{\partial \psi}$ and $B_\psi = -\frac{1}{h_1}\frac{\partial A_z}{\partial \eta}$.

Substituting Eq. (1) into the above equation we obtain:

$$B_\eta(\eta,\psi) = \frac{1}{h_2}\sum_{n=1}^{\infty}\left(n\,\mathcal{A}_n \sinh n\eta \cos n\psi - n\,\mathcal{B}_n \cosh n\eta \sin n\psi\right). \tag{2}$$

The field component B_η can be obtained from the (numerically calculated) Cartesian components by $B_\eta = 1/h_1\,(e\,\sinh\eta\cos\psi\,B_x + e\,\cosh\eta\sin\psi\,B_y)$. At the reference ellipse $\eta = \eta_0$, we can formally obtain the Fourier series expansion

$$B_\eta(\eta_0,\psi) = \sum_{n=1}^{\infty}\left(B_n(\eta_0)\sin n\psi + A_n(\eta_0)\cos n\psi\right), \tag{3}$$

where $B_n(\eta_0) = \frac{1}{\pi}\int_0^{2\pi} B_\eta(\eta_0,\psi)\sin n\psi\,d\psi$ for $n = 1,2,3\ldots$. The coefficients in Eqs. (3) and (2) can, however, not be compared as in the case of the circular harmonics, because the metric coefficient h_2 is a function of ψ. To overcome this problem we define the field components $\tilde{B}_\eta$ and $\tilde{B}_\psi$ by the coordinate derivative without metric coefficients[1]: $\tilde{B}_\eta = \frac{\partial A_z}{\partial\psi}$, $\tilde{B}_\psi = \frac{\partial A_z}{\partial\eta}$. $\tilde{B}_\eta$ can then be expressed as

$$\tilde{B}_\eta(\eta,\psi) = \sum_{n=1}^{\infty}\left(n\mathcal{A}_n \sinh n\eta \cos n\psi - n\mathcal{B}_n \cosh n\eta \sin n\psi\right). \tag{4}$$

At the reference ellipse, $\eta = \eta_0$, the field component $\tilde{B}_\eta$ can now be calculated from the Cartesian components by $\tilde{B}_\eta = e\,\sinh\eta\cos\psi\,B_x + e\,\cosh\eta\sin\psi\,B_y$ and expressed as the Fourier series

$$\tilde{B}_\eta(\eta_0,\psi) = \sum_{n=1}^{\infty}\left(\tilde{B}_n(\eta_0)\sin n\psi + \tilde{A}_n(\eta_0)\cos n\psi\right), \tag{5}$$

where $\tilde{A}_n(\eta_0) = \frac{1}{\pi}\int_0^{2\pi}\tilde{B}_\eta(\eta_0,\psi)\cos n\psi\,d\psi$ and $B_n(\eta_0) = \frac{1}{\pi}\int_0^{2\pi}\tilde{B}_\eta(\eta_0,\psi)\sin n\psi\,d\psi$ for $n = 1,2,3\ldots$. Comparing the coefficients in Eqs. (5) and (4) yields $\mathcal{A}_n = \frac{1}{n\sinh n\eta_0}\tilde{A}_n(\eta_0)$, and $\mathcal{B}_n = -\frac{1}{n\cosh n\eta_0}\tilde{B}_n(\eta_0)$. The vector fields (eigenmodes) corresponding to the $\mathcal{B}_n$ $(n = 1,2,3,4)$ multipole components are displayed in Figure 1. The field components in the elliptic aperture domain can therefore be expressed as

$$\begin{aligned}
B_\eta(\eta,\psi) = \frac{1}{h_2}\sum_{n=1}^{\infty}(\tilde{B}_n(\eta_0)\frac{\cosh n\eta}{\cosh n\eta_0}\sin n\psi\,+ \\
\tilde{A}_n(\eta_0)\frac{\sinh n\eta}{\sinh n\eta_0}\cos n\psi), \\
B_\psi(\eta,\psi) = \frac{1}{h_1}\sum_{n=1}^{\infty}(\tilde{B}_n(\eta_0)\frac{\sinh n\eta}{\cosh n\eta_0}\cos n\psi\,- \\
\tilde{A}_n(\eta_0)\frac{\cosh n\eta}{\sinh n\eta_0}\sin n\psi).
\end{aligned} \tag{6}$$

To avoid the calculation of the flux density by numerical differentiation, it is again possible to perform a Fourier series expansion of the vector potential at the reference ellipse: $A_z(\eta_0,\psi = \sum_{n=1}^{\infty}(C_n(\eta_0)\sin n\psi + D_n(\eta_0)\cos n\psi)$. Substituting the expressions for $\mathcal{A}_n$ and $\mathcal{B}_n$ into (1) for $\eta = \eta_0$ and comparing the coefficients yields $\tilde{B}_n(\eta_0) = -n\,D_n(\eta_0)$, and $\tilde{A}_n(\eta_0) = n\,C_n(\eta_0)$. Hence, we can express B_η (and B_ψ) everywhere inside the domain as

$$B_\eta(\eta,\psi) = \frac{1}{h_2}\sum_{n=1}^{\infty}\left(-n\,D_n(\eta_0)\sin n\psi + n\,C_n(\eta_0)\cos n\psi\right), \tag{7}$$

i.e., the magnetic flux density is directly expressed as a function of the multipoles obtained from the series expansion of the vector potential at η_0. Numerical differentiation of the field solutions is avoided, and yet the multipole coefficients have their usual physical meaning as the "dipole type" B_1 etc.

[1]This is the exterior derivative in differential-form calculus.

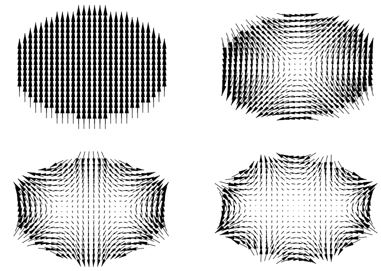

Fig. 1. Vector fields corresponding to the $\mathcal{B}_n$ $(n = 1,2,3,4)$ multipole components in elliptic coordinates (top left to bottom right).

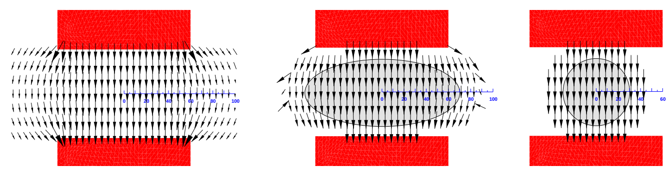

Fig. 2. Left: Numerically calculated field distribution between the poles of a dipole magnet. Middle and right: Field calculated from the truncated series. Middle: Elliptic coordinates (n = 40, a = 70 mm, b = 30 mm). Right: Circular coordinates (n = 40, r_0 = 30 mm).

IV. Results

Fig. 2 (left) shows the numerically calculated field between the poles of a dipole magnet. The field resulting from the truncated series (n=40) is shown in the middle. The reference ellipse has a major-semi axis of 70 mm and a semi-minor axis of 30 mm. Notice how numerical errors dominate the field solution outside of the reference ellipse; all field vectors that deviate by more than 20% in amplitude are omitted from the plot. The advantage of the elliptic multipole expansion is obvious from the comparison to the solution from the truncated series of the circular multipole expansion (right). The effect of the fringe field is better modeled in the elliptic coordinate system and thus the elliptic multipole coefficients are better suited to the optimization of dipole magnets with a large aspect ratio of their air gaps. More types of field harmonics will be presented in the full paper.

References

[1] S. Russenschuck: *Field Computation for Accelerator Magnets*, Wiley-VCH, 2010

[2] Schnizer, P., Schnizer B., Akishin, P., Fischer E.: Theory and application of plane elliptic multipoles for static magnetic fields, Nuclear Instruments and Methods in Physics Research A, Elsevier, 2009